A survey of the Haagerup property

Alain Valette

Nancy, 24 August 2015

Uffe HAAGERUP, 19 December 1949 - 5 July 2015

From Nigel Higson's website: "*Haagerup is a serious, hard analyst; if you give him an inequality and it's true, he'll prove it.*"

1 Affine isometric actions on Hilbert spaces

1.1 Motivations and definitions

G: a locally compact group acting by isometries on a metric space (X, d). Say that the action is (metrically) **proper** if

$$\lim_{g\to\infty} d(x, g.x) = +\infty$$

for every $x \in X$

Definition 1.1. *(Gromov 1990) G is* **a-(T)-menable** *if G admits a proper isometric action on a Hilbert space.*

Recall:

Theorem 1.2. *(Delorme-Guichardet 1972) A locally compact, σ-compact group G has Kazhdan's property (T) if and only if every isometric action on a (affine) Hilbert space has a globally fixed point.*

The case of finite-dimensional Hilbert spaces is settled by Bieberbach's theorem (1910):

Theorem 1.3. *If G is a discrete group acting properly isometrically on Euclidean space $\mathbb{E}^n$, then G fits in a short exact sequence*

$$0 \to \mathbf{Z}^k \to G \to F \to 1,$$

where F is finite and $k \leq n$.

In infinite dimension, we get lots of extra stuff!

1.2 Amenable groups

Theorem 1.4. *(Bekka-Cherix-V. 1993) σ-finite amenable groups are a-(T)-menable.*

Proof for G discrete. Set $G = \cup_n S_n$, an increasing union of finite subsets. By amenability, find a sequence (F_n) of Følner sets:

$$\frac{|S_n F_n \Delta F_n|}{|F_n|} < 2^{-n}.$$

Let $\xi_n \in \ell^2(G)$ be the normalized characteristic function of F_n:

$$\xi_n(g) = \begin{cases} 0 & if \quad g \notin F_n \\ \frac{1}{|F_n|^{1/2}} & if \quad g \in F_n \end{cases}$$

Let λ be the left regular representation of G on $\ell^2(G)$. The isometric action

$$\alpha_n(g)(v) = \lambda(g)v + n(\xi_n - \lambda(g)\xi_n)$$

admits $n\xi_n$ as globally fixed point (so it is not proper).
Idea: pile the α_n's up, so to move the fixed point to infinity.
On $\oplus_{n>0}\ell^2(G)$, define $\alpha(g) = \oplus_{n>0}\alpha_n(g)$, i.e.

$$\alpha(g)(\oplus_n v_n) = \oplus_n[\lambda(g)v_n + n(\xi_n - \lambda(g)\xi_n)]$$

This isometric action is well-defined as $\sum_n n^2\|\xi_n - \lambda(g)\xi_n\|^2 < \infty$ since $\sum_n n^2 2^{-n} < \infty$.

To check it is proper: fix $R > 0$, must prove that $\{g \in G : \|\alpha(g)(0)\| \leq R\}$ is finite.
But $\|\alpha(g)(0)\| \leq R$ implies $n^2\|\xi_n - \lambda(g)\xi_n\|^2 \leq R^2$ for every n. Fix $n \gg 0$ so that $\frac{R^2}{n^2} < 2$. Since $\|\xi_n - \lambda(h)\xi_n\|^2 = 2$ when $hF_n \cap F_n = \emptyset$, this forces $g \in F_nF_n^{-1}$, a finite set. □

1.3 Construction of actions

Definition 1.5. *(Haglund-Paulin) A* **space with walls** *is a pair $(X, \mathcal{W})$ where X is a set and $\mathcal{W}$ is a set of partitions of X into 2 classes, called* **walls***, such that for every $x, y \in X$ the number $w(x, y)$ of walls separating x from y, is finite.*

Observe that $w(.,.)$ defines a pseudo-metric on X - the **wall metric**. A group G of automorphisms of $(X, \mathcal{W})$ acts properly if $\lim_{g\to\infty} w(gx_0, x_0) = \infty$ for some (hence every) $x_0 \in X$.

Example 1. *1. Trees: removing an edge disconnects the tree into 2 half-trees, so we view the set E of edges as a set of walls, for which $w(x, y) = d(x, y)$ (combinatorial distance).*

2. *CAT(0) cube complexes: by a result of Sageev (1994) hyperplanes separate such a complex into 2 connected components. Moreover, for vertices* x, y *we have* $w(x, y) = d_1(x, y)$*, the combinatorial distance in the 1-skeleton.*

Proposition 1.6. *(Haglund-Paulin-V) Every group acting properly on a space with walls, is a-(T)-menable. (In particular, every group acting properly on a tree or on a CAT(0) cube complex, is a-(T)-menable).*

Example 2. *The following groups are a-(T)-menable:*

- *Free groups,* $SL_2(\mathbf{Z})$, $SL_2(\mathbf{Q}_p)$*...*
- *Coxeter groups (Bojezko-Januszkiewicz-Spatzier 1983)*
- *R. Thompson's groups* F, T, V *(Farley 2003)*
- *Groups satisfying the* $C'(1/6)$*-small cancellation condition (Wise 2004).*

Proof of Proposition 1.6: In a space with walls $(X, \mathcal{W})$, define a **half-space** as one class of the partition given by a wall. Let H be the set of half-spaces. For $x \in X$, let χ_x be the characteristic function of the set of half-spaces through x. Observe that $\chi_x - \chi_y$ has finite support, so belongs to $\ell^2(H)$. If G acts properly on X, let π be the permutation representation of G on $\ell^2(H)$, define an isometric action as:

$$\alpha(g)v = \pi(g)(v) + (\chi_{gx_0} - \chi_{x_0})$$

where x_0 is some base-point in X. Then $\|\alpha(g)(0)\|^2 = 2w(gx_0, x_0)$ so that this action α is proper. $\square$

2 Representation theory

Let G be a locally compact group, and π a unitary representation of G on a Hilbert space $\mathcal{H}$.

Definition 2.1. *1. π is C_0 (or:* **mixing***) if, for every $\xi, \eta \in \mathcal{H}$: $\lim_{g\to\infty}\langle\pi(g)\xi|\eta\rangle = 0$;*

2. π **almost has invariant vectors** *if for every $\epsilon > 0$ and every compact subset $K \subset G$, there exists a unit vector $\xi \in \mathcal{H}$ such that*

$$\max_{g\in K} \|\pi(g)\xi - \xi\| < \epsilon;$$

3. G has the **Haagerup property** *if G admits a C_0-representation almost having invariant vectors.*

Remark 2.2. *A finite-dimensional unitary representation* π *cannot be* C_0*. Suppose by contradiction it is* C_0*. From* $1 = |\det \pi(g)|$ *we get:*

$$1 = \lim_{g \to \infty} |\det \pi(g)| = \lim_{g \to \infty} |\sum_{\sigma \in Sym(n)} \epsilon(\sigma) \prod_{i=1}^{n} \langle \pi(g) e_i | e_{\sigma(i)} \rangle| = 0,$$

where $n = \dim \pi$*.*

Example 3.
- *Free groups have the Haagerup property (Haagerup 1979).*
- *For* G *locally compact group, the left regular representation* λ *on* $L^2(G)$ *is always* C_0*; and it has almost invariant vectors if and only if* G *is amenable. So amenable groups have the Haagerup property.*

Theorem 2.3. *A-(T)-menable groups have the Haagerup property. For σ-compact groups, the converse is true.* □

The proof of the converse mimics the proof that amenable groups are a-(T)-menable.

Theorem 2.4. *(Faraut-Harzallah 1971) The groups $O(n, 1)$ and $U(n, 1)$ (and all their closed subgroups) have the Haagerup property.* □

Remark 2.5. *For $U(n, 1)$, an open question is to find a direct, geometric proof of a-(T)-menability.*

3 Why do we care?

3.1 von Neumann algebras

The Haagerup property can be defined for tracial von Neumann algebras; for G discrete, G has the Haagerup property if and only if the group von Neumann algebra $L(G)$ has the Haagerup property (Choda, Jolissaint).

3.2 Baum-Connes conjecture

Theorem 3.1. *(Higson-Kasparov 1997) A-(T)-menable groups satisfy the Baum-Connes conjecture, i.e. the assembly map* $\mu_G : K_i^G(\underline{E}G) \to K_i(C_r^*(G))$ $(i = 0, 1)$ *is an isomorphism.*

Here:

- $\underline{E}G$ is the classifying space for proper actions of G;
- $K_i^G(.)$ is G-equivariant K-homology;
- $C_r^*(G)$ is the reduced C^*-algebra of G (=norm closure of $\lambda(L^1(G))$);
- $K_i(.)$ is analytical K-theory for Banach algebras.

Recall that the Baum-Connes conjecture for discrete group G, implies two famous conjectures:

- (Kaplansky-Kadison conjecture) For G torsion-free, $C_r^*(G)$ has no non-trivial idempotent.
- (Novikov conjecture) The higher signatures are homotopy invariant on smooth manifolds with fundamental group G.

4 Other characterizations

4.1 Infinite-dimensional hyperbolic spaces

Proposition 4.1. *(Gromov) Let G be a second countable, locally compact group. TFAE:*

1. *G is a-(T)-menable;*

2. *G admits a proper isometric action on the infinite-dimensional real hyperbolic space $\mathbb{H}^\infty(\mathbb{R})$;*

3. *G admits a proper isometric action on the infinite-dimensional complex hyperbolic space $\mathbb{H}^\infty(\mathbb{C})$.*

4.2 Spaces with measured walls

Consider $X_n = \mathbb{H}^n(\mathbb{R})$, n-dimensional real hyperbolic space. The isometry group $O(n,1)$ acts transitively on the set $\mathcal{H}$ of hyperplanes, that carries an $O(n,1)$-invariant measure μ. For $a, b \in X_n$, any hyperplane H intersects the segment $[a,b]$ in at most one point, and by **Crofton's formula** we have:

$$\int_{\mathcal{H}} |H \cap [a,b]| \, d\mu(H) = \lambda d(a,b),$$

where $\lambda > 0$ only depends on the normalization of μ.

Since hyperplanes separate X_n into two connected components, it is clear that there is a "continuous" version of spaces with walls underlying this example.

Definition 4.2. *(Cherix-Martin-V. 2003) A* **space with measured walls** *is a 4-tuple* $(X, \mathcal{W}, \mathcal{B}, \mu)$ *where* X *is a set,* $\mathcal{W}$ *is a collection of walls on* X*,* $\mathcal{B}$ *is a* σ*-algebra of sets on* $\mathcal{W}$ *such that, for every* $x, y \in X$*, the set* $W(x|y)$ *of walls separating* x *from* y *belongs to* $\mathcal{B}$*, and* μ *is a measure on* $\mathcal{B}$ *such that* $w(x, y) =: \mu(W(x|y)) < \infty$*.*

Example 4.

- *Spaces with walls (with counting measure)*
- n*-dimensional real hyperbolic space*
- *Real trees (with the measure that gives its length to a segment)*

As for discrete spaces with walls: proper action on a space with measured walls $\Rightarrow$ a-(T)-menability. But now there is a converse.

Theorem 4.3. *(Chatterji-Drutu-Haglund 2010) Let G be a second countable, locally compact group. TFAE:*

1. *G is a-(T)-menable;*
2. *G admits a proper action on a space with measured walls;*
3. *G admits a proper isometric action on some subset of L^p, for some $p \in [1, 2]$.*

The key concept behind that result is median spaces.

Definition 4.4. *A metric space (X, d) is a* **median space** *if, for every $a, b, c \in X$, there exists a unique median point $m = m(a, b, c)$, i.e. a unique point $m \in X$ such that $d(a, m) + d(m.b) = d(a, b)$, and similarly for $\{b, c\}$ and $\{c, a\}$.*

Example 5. • *Trees;*

- $\mathbb{R}^n$ *with the* ℓ^1*-norm.*
- L^1

Theorem 4.5. *(CDH 2010)*

1. *Any space with measured walls embeds isometrically into a (canonical) median space.*
2. *Any median space has a canonical structure of space with measured walls.*
3. *Any median space embeds isometrically into* L^1*.*

5 Permanence properties

A-(T)-menability is preserved under the following group constructions:

- Closed subgroups.
- Finite direct products.
- Increasing union of countably many open subgroups.

For discrete groups, a-(T)-menability is preserved under free products, or amalgamated products over finite groups. More generally:

Proposition 5.1. *(Jolissaint-Julg-V. 2001) Let G be a countable group acting without inversion on a tree, with* **finite** *edge stabilizers. G is a-(T)-menable if and only if every vertex stabilizer is a-(T)-menable.*

Main defect of the Haagerup property: **not** stable under semi-direct products!

Definition 5.2. *Let H be a closed subgroup of the locally compact group G. The pair (G, H) has the* **relative property (T)** *if every isometric G-action on a Hilbert space has an H-fixed point.*

Clearly: if G is a-(T)-menable and (G, H) has relative property (T), then H is compact.

Theorem 5.3. *(Kazhdan 1967) The pairs $(\mathbf{Z}^2 \rtimes SL_2(\mathbf{Z}), \mathbf{Z}^2)$ and $(\mathbf{R}^2 \rtimes SL_2(\mathbf{R}), \mathbf{R}^2)$ have the relative property (T).*

Observe that $\mathbf{Z}^2, SL_2(\mathbf{Z}), \mathbf{R}^2, SL_2(\mathbf{R})$ are a-(T)-menable!

There are however some positive results.

Proposition 5.4. *(JJV 2001) Let H be a closed subgroup of G, with H co-Følner in G (i.e. the quasi-regular representation ρ on $L^2(G/H)$ almost has invariant vectors). If H has the Haagerup property, then so does G.*

Sketch of proof: Let π be a C_0-representation of H almost having invariant vectors. Induce it up to G: the representation $Ind_H^G \pi$ is still C_0 (Bekka), and weakly contains $Ind_H^G 1_H = \rho$. Now ρ weakly contains 1_G, so by transitivity of weak containment $Ind_H^G \pi$ almost has invariant vectors. □

Corollary 5.5. *Let $N \triangleleft G$ be a closed normal subgroup. If N has the Haagerup property and G/N is amenable, then G has the Haagerup property.*

Remark 5.6. *It is an open question to prove directly the above results for a-(T)-menability.*

Open problem: If G acts on W by automorphisms, when is $W \rtimes G$ a-(T)-menable? (Of course, W and G must be a-(T)-menable!)

Example 6. *Let G, H be discrete a-(T)-menable; let $W = \star_G H$ be the free product of copies of H indexed by G, with G acting by shifting indices. Then $W \rtimes G$ is a-(T)-menable, as $W \rtimes G \simeq H \star G$.*

Replace now free product by direct sum: set $W = \oplus_G H$; then $W \rtimes G =: H \wr G$, the **wreath product** of H and G.

Theorem 5.7. *(Cornulier-Stalder-V. 2009) If G, H are discrete, a-(T)-menable groups, than $H \wr G$ is a-(T)-menable.*

The situation is more subtle for **permutational wreath products** $H \wr_Q G =: (\oplus_Q H) \rtimes G$, where Q is a quotient of G. We conjectured in CSV that, if H, G are a-(T)-menable, then so is $H \wr_Q G$. This was false!

Theorem 5.8. *(Chifan-Ioana 2011) Let H, G, Q be countable groups, with H non-trivial and Q a quotient of G. The permutational wreath product $H \wr_Q G$ is a-(T)-menable if and only if H, G and Q are.*

Example 7. *View $SL_3(\mathbf{Z})$ as a quotient of the free group $\mathbb{F}_2$. Then $\mathbf{Z} \wr_{SL_3(\mathbf{Z})} \mathbb{F}_2$ is not a-(T)-menable.*