The quasi-symmetric Hölder equivalence problem

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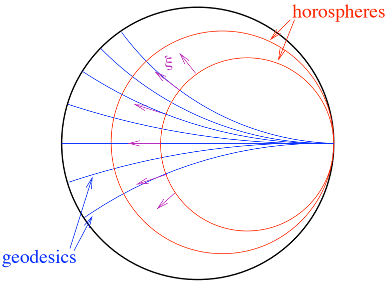
Today's menu: a problem in metric geometry motivated by negative curvature.

Main actor: complex hyperbolic plane $H^2_{\mathbb{C}}$.

The ball $B \subset \mathbb{C}^2$ has biholomorphism group $PU(2,1)$, which preserves a complete Riemannian metric on B and the contact structure of complex tangents on ∂B.
$PU(2,1)$ has a subgroup $S = \mathbb{R} \rtimes Heis$ acting simply transitively on the ball. The conjugation action of $\mathbb{R}$ on $Heis$ is by dilations, $Lie(Heis) = V_1 \oplus V_2$, $\delta_t = e^{jt/4}$ on V_j.
The induced metric on S is of the form $dt^2 + \delta_t^* g_0 = dt^2 + e^{t/2} g_{V_1} + e^t g_{V_2}$, for some left-invariant Riemannian metric g_0 on $Heis$.
$\mathbb{R}$ factors are geodesics, $Heis$-orbits are horospheres. $\frac{\partial}{\partial t}$ and V_2 generate a complex geodesic, curvature -1. Each line in V_1 generates with $\frac{\partial}{\partial t}$ a Lagrangian plane, curvature $-\frac{1}{4}$.



Definition

Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is quasi-isometric to a δ-pinched complete simply connected Riemannian manifold.*

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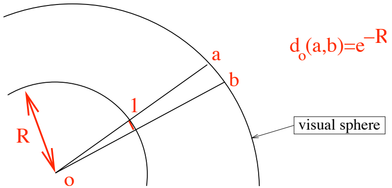
Facts.

- Negatively curved manifolds M have a visual sphere ∂M, equipped with a visual metric.
- If M is δ-pinched, polar coordinates define a $C^{\sqrt{-\ }}$-Hölder homeomorphism from the round sphere $S \to \partial M$, with 1-Lipschitz inverse.
- Quasi-isometries between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.



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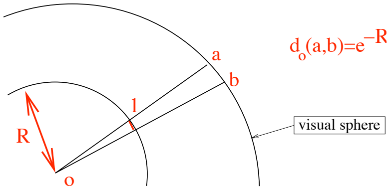

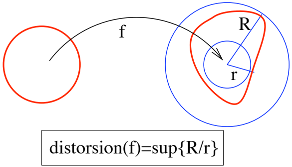

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$\alpha_{qs}(X) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a* C *homeomorphism with Lipschitz inverse from Euclidean space to a metric space quasisymmetric to* $X\}$.

By definition, $\sqrt{-\delta(M)} \leq \alpha_{qs}(\partial M)$.

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Conjecture

$\alpha_{qs}(Heis) = \frac{1}{2}$.

I will explain:

Theorem

$\alpha_{qs}(Heis) \leq \frac{2}{3}$.

Note that consequence $\delta(H^2_{\mathbb{C}}) \geq -\frac{4}{9}$ was already known.

Question (Hölder equivalence problem, Gromov 1993)

Let X be a sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism from Euclidean space to X which is C -Hölder continuous ?

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Theorem (Gromov 1993)

Let metric space X have dimension n, Hausdorff dimension Q. Then $\alpha(X) \leq \frac{n}{Q}$.
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Let X be a $2m+1$-dimensional contact manifold. Then $\alpha(X) \leq \frac{m+1}{m+2}$ $(\leq \frac{2m}{2m+1})$.

Best known bound $\alpha(Heis) \leq \frac{2}{3}$.

Gromov's slogan:

Here
$\mathrm{Euclidean}$
$\exists$ k-dimensional subset with Hausdorff dimension $\leq k$

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Furthermore, only quasisymmetric invariants are used, whence

$$\alpha_{qs}(\text{Heisenberg}) \leq \frac{k}{k'}.$$

Let $f : \mathbb{R}^3 \to Heis$ be a Hölder homeomorphism with Lipschitz $f^{-1} : Heis \to \mathbb{R}^3$. Let $v : \mathbb{R}^3 \to \mathbb{R}$ be a coordinate function. Then $u = v \circ f^{-1}$ is Lipschitz. Imagine that *coarea inequality* holds:

$$\int_G Lip_u^4 \leq \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} Lip_u^3 \right) dt \leq \mathrm{const.} \int_{\mathbb{R}} \mathcal{H}^3(u^{-1}(t))\, dt. \qquad (1)$$

Here, Lip_u denotes the local Lipschitz constant. Since, for non constant u, $\int_X Lip_u^4 > 0$, this shows that there exists $t \in \mathbb{R}$ such that $\mathcal{H}^3(u^{-1}(t)) > 0$, and therefore $u^{-1}(t)$ has Hausdorff dimension at least 3.

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Strategy: replace conformally invariant integrals $\int Lip_u^4$ with *packing measures* which are quasisymmetric invariants and satisfy coarea inequality in the right direction. If possible, extend to vector valued maps u.

Let N be an integer, let $\ell \geq 1$. Let X be a metric space. An (N, ℓ)-*packing* is a countable collection of balls $\{B_j\}$ such that the collection of concentric balls ℓB_j has multiplicity $< N$.

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Let ϕ be a positive function on the set of balls in X. Define

$$\Phi^{p;}_{N;`}(A) = \sup\{\sum_i \phi(B_i)^p \,;\, \{B_i\}\,(N,\ell)\text{−packing of } X, \text{ centered on } A, \text{ of mesh } \leq \epsilon\}.$$

Define the *packing pre-measure associated to* ϕ by

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Example: $\pi(B) = \text{radius}(B)$ leads to "usual" packing measure Π^p and packing dimension. It is Hölder covariant.

Example: let $u : X \to M$ measure space. $e_u(B) = \text{measure}(u(B))$ leads to *p-energy* E^p_u. It is quasisymmetry invariant: if g is quasisymmetric, $\forall \ell$, $\exists \ell'$ such that

$$E^p_{u;N;`}(A) \leq E^p_{u';N;`'}(g(A)).$$

Proposition

Let X be a metric space. Let $u : X \to M$ be a map to a measure space (M, μ). Then

$$\forall p \geq 1, \quad E_u^p(X) \leq \int_M E_u^{p-1}(u^{-1}(m))\, d\mu(m).$$

$$\begin{aligned}
\sum_i \mu(u(B_i))^p &= \sum_i \Big(\int_M 1_{u(B_i)}(m)\, d\mu(m)\Big)\mu(u(B_i))^{p-1} \\
&= \int_M \Big(\sum_i 1_{u(B_i)}(m)\mu(u(B_i))^{p-1}\Big)\, d\mu(m) \\
&= \int_M \Big(\sum_{\{i\,;\, m\in u(B_i)\}} \mu(u(B_i))^{p-1}\Big)\, d\mu(m) \\
&\leq \int_M E_u^{p-1;2} \;\; d\mu(m).\ q.e.d.
\end{aligned}$$

If u is Lipschitz and $\mu(B) \sim \mathrm{radius}(B)^d$ in M, $E_u^{p-1} \leq \Pi^{d(p-1)}$, thus

$$E_u^p > 0 \quad \Rightarrow \quad \exists m, \dim(u^{-1}(m)) \geq d(p-1).$$

Corollary

Pick a submersion $v : \mathbb{R}^3 \to \mathbb{R}^d$*. Assume that for all homeomorphisms* $g : Heis \to \mathbb{R}^3$, $E^{4=d}_{v\circ g} > 0$*. Then* $\alpha_{qs}(G) \leq \dfrac{3-d}{4-d}$.

Corollary

Pick a submersion $v : \mathbb{R}^3 \to \mathbb{R}^d$. Assume that for all homeomorphisms $g : Heis \to \mathbb{R}^3$, $E^{4=d}_{v\circ g} > 0$. Then $\alpha_{qs}(G) \leq \dfrac{3-d}{4-d}$.

Proposition

If $u : G \to \mathbb{R}$ is continuous and non constant, $E^4_u > 0$.

Corollary: $\alpha_{qs}(G) \leq \frac{2}{3}$.

Proof Length-area method. Let Γ denote the family of unit segments parallel to a side of unit n-cube. For all positive functions ρ on the square,

$$\int_0^1 \left(\int \rho\, ds\right) d\gamma \leq \int \rho^n \, dx_1 \dots dx_n.$$

Replace integrals with packing (resp. covering) measures, apply to family of parallel horizontal line segments in *Heis*.

Proposition

Let X be a metric space. Let Γ be a family of subsets of X, equipped with a measure $d\gamma$. For each $\gamma \in \Gamma$, a probability measure m is given on γ. Let $p \geq 1$. Assume that

$$\int_{\{\ \in\Gamma;\ \ \cap B\neq\emptyset\}} m\ (\gamma \cap \ell B)^{1-p}\, d\gamma \leq \tau.$$

Then, for every function ϕ on the set of balls of X,

$$\Phi^p(X) \geq \frac{1}{\tau}\int_\Gamma \tilde{\Phi}^1(\gamma)^p\, d\gamma.$$

Proof Let $1_i(\gamma) = 1$ iff $\gamma \cap B_i \neq \emptyset$. The balls such that $1_i(\gamma) = 1$ cover γ, thus

$$\tilde{\Phi}^{1;}\ (\gamma) \leq \sum_i \phi(B_i)1_i(\gamma) = \sum_i \phi(B_i)1_i(\gamma)m\ (\gamma \cap \ell B_i)^{\frac{1-p}{p}}\, m\ (\gamma \cap \ell B_i)^{\frac{p-1}{p}}.$$

Hölder's inequality gives

$$\tilde{\Phi}^{1;}\ (\gamma)^p \quad \leq \quad \left(\sum_i \phi(B_i)^p 1_i(\gamma) m\ (\gamma \cap \ell B_i)^{1-p}\right)\left(\sum_i m\ (\gamma \cap \ell B_i)\right)^{p-1}.$$

Integrate over Γ.

Apply above Corollary to $d = 2$?

Unfortunately, there exist Lipschitz open maps $u : Heis \to \mathbb{R}^2$ such that $E_u^{4=3} < \infty$, whence $E_u^2 = 0$.

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Note that $E_u^2 = 0 \Rightarrow Du$ is not onto almost everywhere.

Question

Let $g : Heis \to \mathbb{R}^3$ be Lipschitz. Assume that the differential of g has rank ≤ 1 almost everywhere. Show that g cannot be a homeomorphism.