Plateau's problem, isoperimetric inequalities, and large scale geometry

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(joint with Alexander Lytchak)

NPC & ID Nancy: 24 – 28 August 2015

Overview

Part I

Area minimizing discs in metric spaces

(existence and regularity)

Part II

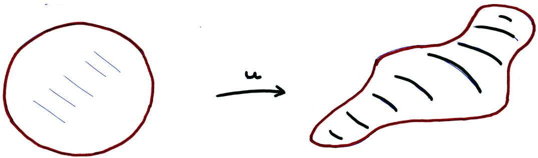
Applications to large scale geometry

(geometry of spaces with quadratic Dehn function)

Part I:

Plateau's problem

To find surface of least area with prescribed boundary



Some solutions:

- Discs in nice Riemannian manifolds (Douglas '31, Radó '30, Morrey '48).
- Surfaces with fixed genus in nice Riemannian manifolds (Courant '37, Jost '85).
- Integral currents in $\mathbb{R}^n$ (Federer-Fleming '60, . . .).

Plateau's Problem in metric spaces

Generalizations:

- Discs in:
  - $\mathrm{CAT}(0)$-spaces (Nikolaev '79).
  - some Alexandrov spaces (Mese-Zulkowski '10).
  - Finsler 3-space (Overath-von der Mosel '14).

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- Integral currents in metric spaces:
  - Cpt metric & some Banach spaces (Ambrosio-Kirchheim '00).
  - Dual Banach and $\mathrm{CAT}(0)$-spaces (W.'05).
  - Non-compact boundaries (Ambrosio-Schmidt '13, W. '14).

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Aim: Existence of area min. discs in proper metric spaces.

Metric space valued Sobolev maps

Let (X, d) be complete metric space, D open unit disc in $\mathbb{R}^2$, $p > 1$.

Definition (Reshetnyak '97, Ambrosio '90)

A map $u\colon D \to X$ is in $W^{1,p}(D, X)$ if

- u measurable and essentially separably valued
- $\exists\, g \in L^p(D)$ such that $\forall \varphi \in \mathrm{Lip}_1(X)$ have $\varphi \circ u \in W^{1,p}(D)$ with $|\nabla(\varphi \circ u)| \leq g$ a.e.

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Equivalent definitions:

- Korevaar-Schoen '93, Jost '94, Heinonen-Koskela-Shanmugalingam-Tyson '01, '15.
- Hajłasz '96: $\exists\, h \in L^p(D)$ s. th. for a.e. $x, y \in D$

$$d(u(x), u(y)) \leq |x - y| \cdot (h(x) + h(y)).$$

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Trace map:

- $\exists\, \bar{u}$ rep. such that $t \mapsto \bar{u}(tv)$ is abs. cont. for a.e. $v \in S^1$.
- The trace of u is defined by

$$\mathrm{tr}(u)(v) := \lim_{t \nearrow 1} \bar{u}(tv)$$

and satisfies $\mathrm{tr}(u) \in L^p(S^1, X)$.

Approximate metric differentiability

Proposition (Kirchheim '94, Karmanova '07)

If $u \in W^{1,p}(D, X)$ then for a.e. $z \in D$ there exists a unique seminorm $\mathrm{md}_z\, u$ on $\mathbb{R}^2$ with

$$\mathrm{ap} - \lim_{v \to 0} \frac{d(u(z+v), u(z)) - \mathrm{md}_z\, u(v)}{|v|} = 0.$$

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- If $X = (\mathbb{R}^n, \|\cdot\|)$ then $\operatorname{md}_z u(\cdot) = \|d_z u(\cdot)\|$.

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Remarks:

- If $X = (\mathbb{R}^n, \|\cdot\|)$ then $\operatorname{md}_z u(\cdot) = \|d_z u(\cdot)\|$.
- u is called Q-quasi-conformal if for a.e. z and all $v, w \in S^1$

$$\operatorname{md}_z u(v) \leq Q \cdot \operatorname{md}_z u(w).$$

- X has property (ET) if $\operatorname{md}_z u$ is (deg.) inner product for all u.

  Ex: (Sub-)Riem. mfds, spaces of bounded curvature, etc.

Area of Sobolev maps

Definition

The parametrized Hausdorff area of $u \in W^{1,2}(D, X)$ is

$$\text{Area}(u) = \int_D \mathbf{J}_2(\text{md}_z\, u)\, dz$$

where $\mathbf{J}_2(\|\cdot\|)$ is Hausdorff measure w.r.t. $\|\cdot\|$ of Eucl. unit square.

Remarks:

- If u is injective and Lipschitz then $\text{Area}(u) = \mathcal{H}^2(u(D))$.
- If u has Lusin's property (N) then

$$\text{Area}(u) = \int_X \#\{z : u(z) = x\}\, d\mathcal{H}^2(x).$$

Solution to Plateau's problem

Given $\Gamma \subset X$ Jordan curve let

$$\Lambda(\Gamma) = \{v \in W^{1,2}(D, X) : \text{ tr}(v) \text{ weakly mon. param. of } \Gamma\}.$$

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Theorem (Lytchak-W. '15)

If X is proper and $\Gamma \subset X$ with $\Lambda(\Gamma) \neq \emptyset$ then there exists $u \in \Lambda(\Gamma)$ with

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Remarks:

- $\sqrt{2}$-q.c. is optimal in general.
- If X has property (ET) then obtain u conformal.
- Hausdorff area can be replaced by any quasi-convex area def.

Energies

Reshetnyak's energy:

$$E_+^2(u) = \int_D \mathcal{I}_+^2(\mathrm{md}_z\, u)\, dz,$$

where

$$\mathcal{I}_+^2(\|\cdot\|) = \max\{\|v\|^2 : v \in S^1\}.$$

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Korevaar-Schoen's energy:

$$E^2(u) = \int_D \mathcal{I}_{\mathrm{avg}}^2(\mathrm{md}_z\, u)\, dz,$$

where

$$\mathcal{I}_{\mathrm{avg}}^2(\|\cdot\|) = \frac{1}{\pi}\int_{S^1} \|v\|^2\, dv.$$

Energy minimizers

Theorem (Lytchak-W. '15)

If X is complete and $u \in W^{1,2}(D, X)$ is such that

$$E_+^2(u) \leq E_+^2(u \circ \psi)$$

for every biLipschitz homeomorphism ψ of D then u is $\sqrt{2}$-q.c.

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- $\sqrt{2}$-q.c. is optimal in general.
- If X has property (ET) then obtain u conformal.
- The classical proof

$$0 = \frac{d}{dt} E^2(u \circ \psi_t) = \dots$$

works when X has (ET) but breaks down in general.

Main steps in proof:

1. If a seminorm $\|\cdot\|$ satisfies $\mathcal{I}_+^2(\|\cdot\|) \leq \mathcal{I}_+^2(\|\cdot\| \circ T)$ for all $T \in \mathrm{SL}_2(\mathbb{R})$ then $\|\cdot\|$ is $\sqrt{2}$-q.c.

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2. $T \in \mathrm{SL}_2(\mathbb{R})$, $z \in \mathbb{R}^2$, $r > 0 \Rightarrow \exists\, \rho$ biLip. homeo. of $\mathbb{R}^2$ s.th.
   - ρ conformal on $\mathbb{R}^2 \setminus \overline{B}(z, r)$.
   - $\rho(w) = z + T^{-1}(w - z)$ for $w \in B(z, r)$.

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   - $\rho(w) = z + T^{-1}(w - z)$ for $w \in B(z,r)$.

3. Set $\psi = \rho^{-1} \circ \varphi$, where $\varphi \colon D \to \rho(D)$ conf. diffeo. Then

$$E_+^2(u) - E_+^2(u \circ \psi)$$

$$= \int_{B(z,r)} \left(\mathcal{I}_+^2(\mathrm{md}_w\, u) - \mathcal{I}_+^2(\mathrm{md}_w\, u \circ T)\right) dw$$

$$\simeq \left(\mathcal{I}_+^2(\mathrm{md}_z\, u) - \mathcal{I}_+^2(\mathrm{md}_z\, u \circ T)\right) \cdot |B(z,r)|$$

Back to Plateau's problem

The classical proof in $\mathbb{R}^n$ relies on:

1. Existence of energy minimizers in $\Lambda(\Gamma)$.
2. Energy minimizers are conformal and minimize area.

Problem: Step 2 breaks down in metric spaces.

Back to Plateau's problem

The classical proof in $\mathbb{R}^n$ relies on:

1. Existence of energy minimizers in $\Lambda(\Gamma)$.
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Problem: Step 2 breaks down in metric spaces.

Proposition (Lytchak-W. '15)

There exists X biLipschitz homeo. to S^2 and $\Gamma \subset X$ Jordan such that energy minimizers in $\Lambda(\Gamma)$ are not area minimizers.

However:

- If X has (ET) then energy minimizers are area minimizers.

Main ingredients in proof:

1. If $(u_n) \subset \Lambda(\Gamma)$ has bounded energy and satisfies 3-pt condition then $\exists\, u_{n_j}$ such that $u_{n_j} \to u$ in L^2 for some $u \in \Lambda(\Gamma)$.

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2. If (u_n) has bounded energy and $u_n \to u$ in L^2 then

$$\operatorname{Area}(u) \leq \liminf_{n\to\infty} \operatorname{Area}(u_n).$$

Works for any quasi-convex area def.

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3. If $u \in \Lambda(\Gamma)$ then $\exists\, v \in \Lambda(\Gamma)$ which is $\sqrt{2}$-q.c and

$$\text{Area}(v) \leq \text{Area}(u)$$

and hence $E_+^2(v) \leq 2\,\text{Area}(u)$.

Local quadratic isoperimetric inequality

Definition

A metric space X admits local quadratic isoperimetric inequality if $\exists C, r > 0$ such that for every $c\colon S^1 \to X$ Lipschitz with $\ell(c) \leq r$ there exists $u \in W^{1,2}(D, X)$ with $\mathrm{tr}(u) = c$ and

$$\mathrm{Area}(u) \leq C \cdot \ell(c)^2.$$

Examples:

- Homogeneously regular Riemannian manifolds (Morrey).
- Compact Lipschitz manifolds.
- $\mathrm{CAT}(\kappa)$-spaces and compact Alexandrov spaces.
- Banach spaces.
- Some Carnot-Carathéodory spaces (e.g. Heisenberg $\mathbb{H}^{n\geq 2}$).

Regularity of area minimizers

Theorem (Lytchak-W. '15)

Let X be complete metric space, $\Gamma \subset X$ Jordan curve, $u \in \Lambda(\Gamma)$ such that

$$\text{Area}(u) = \inf\{\text{Area}(v) : v \in \Lambda(\Gamma)\}$$

and u is Q-quasi-conformal. If X admits loc. quad. isop. ineq. then:

1. *u is continuous and has Lusin (N), in fact u is loc. $W^{1,p>2}$.*
2. *u is locally α-Hölder and extends cont. to $\overline{D}$, $\alpha = \frac{1}{4\pi Q^2 C}$.*
3. *If Γ is chord-arc then u is Hölder on $\overline{D}$.*

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Remarks:

- Proof along lines of classical proofs.
- Hölder exponent α is optimal (when $Q = 1$).
- If X proper geod. and isop. constant $C < \frac{1}{8\pi}$ then X real tree.

Part II:

Applications to large scale geometry

Quadratic isoperimetric inequality

Definition

X admits quad. isop. ineq. with constant C if every $c\colon S^1 \to X$ Lipschitz has $u \in W^{1,2}(D, X)$ with $\mathrm{tr}(u) = c$ and

$$\mathrm{Area}(u) \leq C \cdot \ell(c)^2.$$

Examples:

- X $\mathrm{CAT}(0)$, Busemann npc, Banach, higher Heisenberg, ...
- X universal covering of closed Riem. mfd. such that $G \curvearrowright X$ geometrically with quadratic Dehn function $\delta_G(n)$.

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Theorem (Papasoglu '96)

If G has quadratic Dehn function then every asymptotic cone of G is simply connected.

Theorem (Lytchak-W. '15)

If X proper, geodesic has quad. isop. inequality then so does every asymptotic cone of X.

Remarks:

- Obtain α-Hölder, p-Sobolev fillings, where $\alpha > 0$, $p > 2$ only depend on isoperimetric constant.

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- W. '11: asymptotic cones of X admit homological quad. isop.

Asymptotic cones

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Corollary

If X as above then every asymptotic cone of X

1. *is simply connected.*
2. *has β-Hölder extension property for $\beta \leq \alpha$.*

Gromov hyperbolicity

Theorem (W. '08)

X geodesic metric space. If there exist $\varepsilon, r > 0$ such that every Lip. loop c in X of length $\geq r$ bounds Lip. disc with

$$\text{Area} \leq \frac{1-\varepsilon}{4\pi} \cdot \ell(c)^2$$

then X is Gromov hyperbolic.

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Remarks:

- Gromov obtained theorem with constant $\frac{1}{4000}$.
- Borderline case $\varepsilon = 0$ see A. Lytchak's talk.
- Proof relies on Ambrosio-Kirchheim currents and compactness.
- If X proper then proof without currents.

Proof using Sobolev maps: (Lytchak-W. '15)

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2. If X_ω not metric tree then $\exists\, c$ Jordan Lipschitz loop in X_ω.

   $\Rightarrow \exists$ filling $u \in W^{1,2}(D, X)$ of c.

   $\Rightarrow \text{Area}(u) > 0$.

   $\Rightarrow X_\omega$ contains almost isometric copy of 2-dim. normed ball.

   $\Rightarrow$ contradiction with isoperimetric constant $\frac{1-\varepsilon}{4\pi}$ since

   normed planes have isop. constant $\geq \frac{1}{4\pi}$.

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3. X_ω arbitrary $\Rightarrow X$ Gromov hyperbolic.

Nilpotent groups

Question: Does every finitely generated nilpotent G have

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Theorem (W. '11)

There exists G nilpotent of step 2 such that

$$n^2\varrho(n) \preceq \delta_G(n) \preceq n^2 \log(n)$$

for some function $\varrho(n) \to \infty$ as $n \to \infty$.

Remarks:

- Proof uses homological quad. isop. in asymptotic cones.
- May use quadratic isoperimetric inequality above instead.